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Note

On Multiply Critically h-Connected Graphs

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A conjecture of Slater states that K_{h+1} is the unique k-critically h-connected noncomplete graph for $2k > h$. We prove here that there is no k-critically h-connected graph with order $\geqslant h + k - 2$ for $2k > h + 1$. We prove also that there is no k-critically h-connected line graph for $2k > h$. The last result was conjectured by Maurer and Slater. We apply in our proofs a method introduced by Mader.

1. INTRODUCTION

We use the terminology of Berge [1]. A graph $G = (V, E)$ is said to be *k-critically h-connected* (or simply an (h, k)-graph) if for every $S \subset V$ with $0 \leqslant |S| \leqslant k$, $\kappa(G_{V-S}) = h - |S|$. This notion is introduced by Maurer and Slater in [7]. The following conjecture is proposed [2, 7].

CONJECTURE (Slater). *The complete graph K_{h+1} is the unique k-critically h-connected graph for $2k > h$.*

The above conjecture is verified for $h \leqslant 6$ by Maurer and Slater [7] and for $h \leqslant 10$ by Mader [5, 6].

We prove in this paper that there is no k-critically h-connected graph with order $\geqslant h + k - 2$ for $2k > h + 1$. This implies that a counterexample to Slater's conjecture, if it exists, must of relatively small order for even h. Our methods can be used to prove Slater's conjecture for even h not exceeding 18.

We prove also that for $2k > h$, there is no (h, k)-noncomplete line graph, answering a conjecture of Maurer and Slater [8].

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2. Atomic Sequence of a Graph

Let $G = (V, E)$ be a connected noncomplete graph and F be a subset of V. The subset F is said to be a fragment of G if $|N(F)| = \kappa(G)$ and $\bar{F} \neq \varnothing$, where $\bar{F} = V - (F \cup \Gamma(F))$ and $N(F) = \Gamma(F) - F$. We see easily that $\bar{F}$ is also a fragment of G. A fragment of minimal cardinality is called an atom. A fundamental property of atoms is the following.

Theorem A (Mader [4]). *Let G be a connected noncomplete graph, A be an atom of G and T be a minimum cutset of G. If $A \cap T \neq \varnothing$, then $A \subset T$ and $|A| \leqslant \frac{1}{2}\kappa(G)$.*

Let $G = (V, E)$ be a graph and $\{A_i;\ 1 \leqslant i \leqslant r\}$ be a family of subsets of V. We say that $\{A_i;\ 1 \leqslant i \leqslant r\}$ is an *atomic sequence* of G if A_i is an atom of $G_{V - \bigcup_{j<i} A_j}$; $1 \leqslant i \leqslant r$. We can verify easily using Mader's Theorem A that every k-critically h-connected noncomplete graph has an atomic sequence of length $k+1$ (cf. [3]). We verify also that the elements of this atomic sequence are fragments of G. Therefore $|A_1| \leqslant |A_2| \leqslant \cdots |A_r|$, for any atomic sequence $\{A_i;\ 1 \leqslant i \leqslant r\}$. The proof of these two results is a direct application of the definitions and of Theorem A.

Proposition 2.1. *Let F and F' be two fragments of a noncomplete connected graph $G = (V, E)$. If $G_{N(F) - N(F')}$ is connected, then $F \cap N(F') = \varnothing$ or $\bar{F} \cap N(F') = \varnothing$.*

Proof. Put $N(F) = T$ and $N(F') = T'$. Let C be a connected component of $G_{V-T'}$ such that $C \cap (T - T') = \varnothing$ (such a component exists since $G_{T-T'}$ is a connected subgraph of $G_{V-T'}$). Clearly G_C is a connected subgraph of G_{V-T}. Hence $C \subset F$ or $C \subset \bar{F}$. In the case $C \subset F$, we have $T' = N(C) \subset F \cup N(F)$. Therefore $T' \cap \bar{F} = \varnothing$. Similarly $T' \cap F = \varnothing$ in the case $C \subset \bar{F}$.

Let $\{A_i;\ 1 \leqslant i \leqslant k\}$ be an atomic sequence of a graph $G = (V, E)$. We take $B_j = \bigcup_{i \leqslant j} A_i$; $1 \leqslant j \leqslant k$.

Lemma 2.2 [3]. *Let $G = (V, E)$ be a k-critically h-connected graph and $\{A_i;\ 1 \leqslant i \leqslant k+1\}$ be an atomic sequence of G. Then $N(A_i) \supset B_{i-1}$; $1 \leqslant i \leqslant k+1$. Moreover we have $2|A_k| \leqslant h - |B_{k-1}|$.*

The proof of this lemma is an application of Mader's Theorem A.

Proposition 2.3. *Let $G = (V, E)$ be a noncomplete (h, k)-graph and $\{A_i;\ 1 \leqslant i \leqslant k\}$ be an atomic sequence of G. Then the subgraph spanned by B_j is $(j-1)$-connected; $1 \leqslant j \leqslant k$.*

Proof. Suppose the proposition false and let j be the smallest value for which this occurs. We have $j \geqslant 2$, since an atom is connected (cf. Mader

[4]). Clearly $|B_j| \geqslant j$, hence G_{B_j} has a cutset S such that $|S| \leqslant j-2$. We prove the following.

(1) $A_j \cap S \neq \varnothing$.

Suppose the contrary. Then $S \subset B_{j-1}$. But G_{A_j} is connected (observe that A_j is an atom of the graph spanned by $V - B_{j-1}$, but an atom is connected (Mader [4])). Using Lemma 2.2 we have $N(A_j) \supset B_{j-1} \supset B_{j-1} - S$. The above two relations imply that the subgraph induced by $B_j - S = A_j \cup (B_{j-1} - S)$ is connected, which is a contradiction.

(2) By (1) and the minimality of j, the subgraph induced by $B_{j-1} - S$ is connected. It follows that A_j contains one component P of the subgraph spanned by $B_j - S$. Therefore $N(P) \subset S \cup (N(A_j) - B_{j-1})$. By Lemma 2.2, we have $B_{j-1} \subset N(A_j)$. It follows that $h \leqslant |N(P)| \leqslant |S| + h - |B_{j-1}|$. Hence $|S| \geqslant |B_{j-1}| \geqslant j-1$. This contradiction proves the proposition.

THEOREM 2.4. *Let k and h be two natural numbers such that $k > [h/2]$. Then there is no k-critically h-connected graph with order $\geqslant h+k-2$.*

Proof. Suppose the contrary and let $G = (V, E)$ be an (h, k)-graph such that $|V| \geqslant h+k-2$. Consider an atomic sequence $\{A_i;\ 1 \leqslant i \leqslant k+1\}$ of G. Put $T_i = N(A_i)$; $1 \leqslant i \leqslant k+1$ and $R = T_{k+1} - B_k$. By Lemma 2.2, we have $|R| \leqslant h-k \leqslant k-2$. Let $x \in A_{k+1}$ and $y \in \bar{A}_{k+1}$. Therefore $|R \cup \{x, y\}| \leqslant k$. We prove the following.

(1) $|R \cup (V - T_{k+1})| \geqslant k$.

This is true if $|R| \geqslant 2$, as $|V| \geqslant h+k-2$. Suppose $|R| \leqslant 1$. The last part of Lemma 2.2 can be written $|A_k| \leqslant |R|$. Hence $|A_k| = 1$. It follows that $|B_k| = k$. Therefore $|R| = h-k = 1$, which is a contradiction since K_{k+2} is the unique k-critically $(k+1)$-connected graph (cf. Maurer and Slater [7]).

(2) Let S be a subset of $R \cup (V - T_{k+1})$ such that $|S| = k$ and $S \supset R \cup \{x, y\}$ (such a subset exists by (1) and the relation $|R \cup \{x, y\}| \leqslant k$). Let T be a minimum cutset of G containing S. By Proposition 2.1 and since $T \cap A_{k+1} \neq \varnothing$ and $T \cap \bar{A}_{k+1} \neq \varnothing$, the subgraph induced by $T_{k+1} - T$ is not connected. But $T_{k+1} - T \subset B_k$. Using Proposition 2.3, we have $|T_{k+1} - T| \leqslant |B_k| - k + 1$. Therefore

$$k - |R| \leqslant |T - T_{k+1}| = |T_{k+1} - T| \leqslant |B_k| - k + 1 \qquad (|T| = |T_{k+1}|).$$

It results that $2k - 1 \leqslant |B_k| + |R| = h$. Thus $k \leqslant [h/2]$, which is a contradiction. This contradiction proves the theorem.

CONJECTURE 2.5. *There is no noncomplete (h, k)-graph for $k > [h/2]$.*

This conjecture is weaker than Slater's conjecture for odd h. The two conjectures coincide for even h. We can see easily using Theorem 2.4, that conjecture 2.5 is false if and only if there is a $(2p, p+1)$-graph $G=(V, E)$ such that $2p+4 \leqslant |V| \leqslant 3p-2$ (take a counterexample of minimal cardinality and observe that the deletion of a vertex from a k-critically h-connected graph gives an $(h-1, k-1)$-graph; observe also that this counterexample must be of order not less than $h+4$, otherwise $|A_{k+1}|=1$ and hence $|R|=h-k$, using the notations of Theorem 2.4). In particular to prove Slater's conjecture for even h, it is sufficient to prove it for a graph $G=(V, E)$ such that $h+4 \leqslant |V| \leqslant \lfloor \frac{3}{2}h \rfloor - 2$.

Remark. The above methods can be adapted to prove Slater's conjecture for even h not exceeding 18. Such a proof contains a tedious examination of cases.

3. k-Critically h-Connected Line Graphs

Maurer and Slater showed in [8] that the connectivity of the line graph of a graph is related to its separation into nontrivial components. They formulate Slater's conjecture for the case of a line graph. We will prove this using Proposition 2.1 and the following result.

THEOREM B (Entringer and Slater [2]). *Let k and h be two nonnull natural numbers such that $k > \lfloor h/2 \rfloor$. Then every k-critically h-connected graph contains a vertex of degree h.*

We note that this theorem is a consequence of Lemma 2.2. We proved in [3] that there are at least two such vertices, answering a conjecture of Entringer and Slater [2].

LEMMA 3.1. *Let H be a line graph and x be a vertex of H. Then $N(x)$ can be covered by two cliques of H.*

The proof of this lemma is easy.

THEOREM 3.2. *There is no k-critically h-connected line graph for $k > \lfloor h/2 \rfloor$.*

Proof. Suppose the contrary and let H be a k-critically h-connected line graph. By Theorem B, H contains a vertex x of degree h. By Lemma 3.1, $N(x)$ contains a clique of cardinality not less than $h/2$. Let C be such a clique. We have $|N(x)-C| < k$. Let T be a minimum cutset of H containing $(N(x)-C) \cup \{x\}$. As $N(x)-T$ is connected (it is contained in C), we have $T \cap \{\bar{x}\} = \varnothing$ (observe that $\{x\}$ is a fragment), using Proposition 2.1.

Therefore $N(x) - T$ consists of a unique vertex c. But c is connected to each component of $H_{V-N(x)}$, where V is the vertex-set of H. This contradicts the fact that T is a cutset. This contradiction proves the theorem.

Remark. Theorem 3.2 is equivalent to conjecture 3.5 [8].

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